Problem 1.32

Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2yz^3$, the points $\mathbf{a} = (0, 0, 0)$, $\mathbf{b} = (1, 1, 1)$, and the three paths in Fig. 1.28:

(a) $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$;

(b) $(0, 0, 0) \to (0, 0, 1) \to (0, 1, 1) \to (1, 1, 1)$;

(c) the parabolic path $z = x^2$, $y = x$.

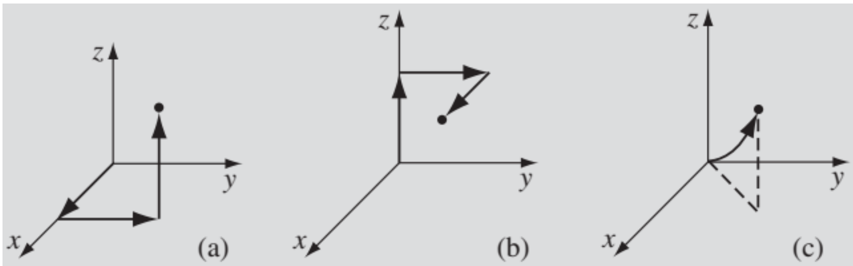

Fig. 1.28

Solution

The aim here is to verify the fundamental theorem for line integrals, which states that

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}),$$

for the three paths shown above. In each case the right side is

$$\begin{aligned} T(1,1,1) - T(0,0,0) &= [(1)^2 + 4(1)(1) + 2(1)(1)^3] - [(0)^2 + 4(0)(0) + 2(0)(0)^3] \\ &= 7. \end{aligned}$$

Part (a)

Evaluate the left side for the specific path: $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$.

$$\int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \nabla T \cdot d\mathbf{l} = \int_{\langle 0,0,0\rangle}^{\langle 1,0,0\rangle} \nabla T \cdot d\mathbf{l} + \int_{\langle 1,0,0\rangle}^{\langle 1,1,0\rangle} \nabla T \cdot d\mathbf{l} + \int_{\langle 1,1,0\rangle}^{\langle 1,1,1\rangle} \nabla T \cdot d\mathbf{l}$$

Along the first line segment, the variation is solely over x while $y = 0$ and $z = 0$; along the second line segment, the variation is solely over y while $x = 1$ and $z = 0$; and along the third line segment, the variation is solely over z while $x = 1$ and $y = 1$.

$$\begin{aligned} \int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \nabla T \cdot d\mathbf{l} &= \int_0^1 (\nabla T)_x \bigg|_{\substack{y=0\\z=0}} dx + \int_0^1 (\nabla T)_y \bigg|_{\substack{x=1\\z=0}} dy + \int_0^1 (\nabla T)_z \bigg|_{\substack{x=1\\y=1}} dz \\ &= \int_0^1 \frac{\partial T}{\partial x}\bigg|_{\substack{y=0\\z=0}} dx + \int_0^1 \frac{\partial T}{\partial y}\bigg|_{\substack{x=1\\z=0}} dy + \int_0^1 \frac{\partial T}{\partial z}\bigg|_{\substack{x=1\\y=1}} dz \end{aligned}$$

Compute the integrals.

$$\begin{aligned}
\int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \nabla T \cdot d\mathbf{l} &= \int_0^1 (2x+4y)\bigg|_{\substack{y=0\\z=0}} dx + \int_0^1 (4x+2z^3)\bigg|_{\substack{x=1\\z=0}} dy + \int_0^1 (6yz^2)\bigg|_{\substack{x=1\\y=1}} dz \\
&= \int_0^1 2x\,dx + \int_0^1 4(1)\,dy + \int_0^1 6(1)z^2\,dz \\
&= 1+4+2 \\
&= 7
\end{aligned}$$

The fundamental theorem is verified.

Part (b)

Evaluate the left side for the specific path: $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$.

$$\int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \nabla T \cdot d\mathbf{l} = \int_{\langle 0,0,0\rangle}^{\langle 0,0,1\rangle} \nabla T \cdot d\mathbf{l} + \int_{\langle 0,0,1\rangle}^{\langle 0,1,1\rangle} \nabla T \cdot d\mathbf{l} + \int_{\langle 0,1,1\rangle}^{\langle 1,1,1\rangle} \nabla T \cdot d\mathbf{l}$$

Along the first line segment, the variation is solely over z while $x=0$ and $y=0$; along the second line segment, the variation is solely over y while $x=0$ and $z=1$; and along the third line segment, the variation is solely over x while $y=1$ and $z=1$.

$$\begin{aligned}
\int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \nabla T \cdot d\mathbf{l} &= \int_0^1 (\nabla T)_z\bigg|_{\substack{x=0\\y=0}} dz + \int_0^1 (\nabla T)_y\bigg|_{\substack{x=0\\z=1}} dy + \int_0^1 (\nabla T)_x\bigg|_{\substack{y=1\\z=1}} dx \\
&= \int_0^1 \frac{\partial T}{\partial z}\bigg|_{\substack{x=0\\y=0}} dz + \int_0^1 \frac{\partial T}{\partial y}\bigg|_{\substack{x=0\\z=1}} dy + \int_0^1 \frac{\partial T}{\partial x}\bigg|_{\substack{y=1\\z=1}} dx \\
&= \int_0^1 (6yz^2)\bigg|_{\substack{x=0\\y=0}} dz + \int_0^1 (4x+2z^3)\bigg|_{\substack{x=0\\z=1}} dy + \int_0^1 (2x+4y)\bigg|_{\substack{y=1\\z=1}} dx \\
&= \int_0^1 6(0)z^2\,dz + \int_0^1 [4(0)+2(1)^3]\,dy + \int_0^1 [2x+4(1)]\,dx \\
&= 0+2+5 \\
&= 7
\end{aligned}$$

The fundamental theorem is verified.

Part (c)

Evaluate the left side for the line integral over the parabolic path parameterized by $\mathbf{l}(t) = \langle t, t, t^2 \rangle$, where $0 \leq t \leq 1$.

$$\begin{aligned}
\int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \nabla T \cdot d\mathbf{l} &= \int_0^1 \nabla T(\mathbf{l}(t)) \cdot \mathbf{l}'(t)\, dt \\
&= \int_0^1 \langle 2(t) + 4(t), 4(t) + 2(t^2)^3, 6(t)(t^2)^2 \rangle \cdot \langle 1, 1, 2t \rangle\, dt \\
&= \int_0^1 \langle 6t, 4t + 2t^6, 6t^5 \rangle \cdot \langle 1, 1, 2t \rangle\, dt \\
&= \int_0^1 [(6t)(1) + (4t + 2t^6)(1) + (6t^5)(2t)]\, dt \\
&= \int_0^1 (10t + 14t^6)\, dt \\
&= 10\left(\frac{1}{2}\right) + 14\left(\frac{1}{7}\right) \\
&= 5 + 2 \\
&= 7
\end{aligned}$$

The fundamental theorem is verified.